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On the Spectrum of Neutron Transport Equation in Finite Bodies

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1. INTRODUCTION

The time dependent neutron transport equation has become the object of a growing interest since, in 1954, Lehner and Wing discovered some unsuspected features of the solution of an initial value problem, relating to the particular case of the one velocity isotropic scattering transport equation in a bare slab [1, 2]. These authors showed that, contrary to the results of elementary diffusion theory, the eigenvalue spectrum of this equation consists of a finite, nonvoid set of real eigenvalues plus a continuous spectrum occupying one half of the spectral plane. In consequence, only a finite number of eigenfunctions exist, and the expansion of the solution of the initial value problem into these eigenfunctions must contain an additional term, in the form of a Cauchy integral evaluated along a vertical path just to the right of the continuous spectrum. This "error" term decays faster than $e^{-v_0 t/l}$, where v_0 is the neutron speed, l the total mean free path, and t the time. An intuitive interpretation of this phenomenon is that any point of the slab experiences a flux which is partly due to neutrons which have undergone only a fixed, finite number of collisions, e.g., one collision, however large may be the time t elapsed after their injection into the slab at $t = 0$: it only suffices for this that the direction of their last flight be fairly parallel to the sides of the slab. It is reasonable to suppose that the contribution of such neutrons will not decay according to a denumerable infinity of discrete modes, but will give rise to a continuous infinity of transients all decaying faster than $e^{-v_0 t/l}$, this limit being reached only by uncollided neutrons which are moving exactly parallel to the sides of the slab.

The extension to n velocity groups [3] shows the same essential features as the one velocity case.

A completely different result has been obtained by K. Jörgens [4] for an arbitrarily shaped but finite body. This author showed that the spectrum of the velocity dependent transport equation is in this case purely discrete, provided that the set of the admissible neutron speeds is bounded away from zero. This result holds also for the multigroup approximation. K. Jörgens, however, did not obtain a true eigenfunction expansion for the solution of the initial value problem, but limited himself to an asymptotic expansion. It has been shown by other workers [5, 6] that an essential difficulty is encountered when looking for the proper (nonasymptotic) expansion, namely, the fact that we know very little about the asymptotic distribution of the point spectrum (i.e., the distribution of the eigenvalues with a large negative real part). Such knowledge is indeed a necessary tool for the proof of the convergence of the eigenfunction expansion. Moreover, it appears that some theorems about the representation of the time dependent solution by means of a Laplace inversion formula cannot avoid certain restrictions upon the time variable, namely, that $t > t_0$, where t_0 is an upper bound for the time that a neutron can spend in the body, without undergoing collisions or escaping from the boundaries. Only if $t > t_0$ may one expect to be able to write the solution in the form of an eigenfunction expansion, and this is in conformity with the limiting case of an infinite body ($t_0 = \infty$), in which the eigenfunctions form an incomplete set, and an error term must always be added to the expansion.

All these studies, very important from the theoretical point of view, seem to have had little influence on the applied reactor works, until R. S. Nelkin [7] observed that the theory of pulsed neutron experiments involves the solution of an initial value problem relating to a finite sample, but, contrary to Jörgens case, without any lower bound for the speed of the neutrons moving in it, since thermal neutrons can have all the energies between zero and infinity. Thus, no upper bound can be imposed to the time required by a neutron to traverse the sample without collisions, and the problem becomes similar, by this respect, to that of Lehner and Wing. By considering a homogeneous sphere, and using a very simplified scattering model, Nelkin was able to prove that the largest eigenvalue of the transport equation, say λ_0, must satisfy the following inequality:

$$\lambda_0 \geqslant - \lim_{v \to 0} \frac{v}{l(v)}$$

just as in the Lehner and Wing problem, where the region $\operatorname{Re} \lambda \leqslant - v_0/l$ was occupied by the continuous spectrum. There is, however, an essential difference: λ_0 does not necessarily exist for all sample sizes. Although Nelkin has not given a rigorous proof of this fact, he has made clear on an intuitive basis that, if the radius a of the sphere is smaller than some characteristic radius a_0, then no eigenvalue exists. Now, this is of great importance from a practical point of view, since it is well known that the pulsed experiments are substantially directed to the measure of the decay constant $-\lambda_0$ of the fundamental mode of the flux after a neutron burst. The nonexistence of this fundamental mode would make pulsed experiments meaningless for samples of very small size.

Nelkin's results have been confirmed by Corngold [8], who has also given a more complete analysis of the eigenvalues of the transport equation. In Corngold's work, however, the spatial dependence of the flux has a prescribed behavior (of the type $e^{i\mathbf{B}\cdot\mathbf{r}}$, where $\mathbf{B}$ is a vector independent of the neutron speed v), and some proofs still retain an heuristic character.[1]

In this paper we try to give a general and rigorous theory of the spectrum of the transport equation for thermal neutrons. We adopt the free gas model, and consider an arbitrary (well shaped) finite body as domain for the space variables. The only important assumption is that of isotropic scattering in the laboratory system. Our results confirm those of Nelkin and Corngold and prove that the spectrum is made of a continuous part plus a finite number of real eigenvalues, this number being zero if the sample is small enough (Main Theorem, Section 8).

The fact that experimental results seem at present to be difficult to interpret regarding the existence of eigenvalues, or even in contradiction with the theory, is probably due to the inadequacy of the experimental techniques. Otherwise, since the problem of the eigenvalue distribution has been treated here in a rigorous manner for the free gas model (and we would hardly expect more sophisticated models to yield fundamentally different results in this respect), we would have to admit that the classical Boltzmann equation itself does not apply to thermal neutron pulses.

2. Preliminary Remarks

Let us consider the time dependent transport equation in an absorbing and scattering homogeneous body [12, 13]:

$$\frac{\partial n(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \operatorname{grad}_{\mathbf{r}} n(\mathbf{r}, \mathbf{v}, t) + v\Sigma_{\mathrm{t}}(v)\, n(\mathbf{r}, \mathbf{v}, t)$$
$$= \int_{\omega} v' \Sigma_{\mathrm{s}}(\mathbf{v}' \to \mathbf{v})\, n(\mathbf{r}, \mathbf{v}', t)\, d\omega' \tag{1}$$

[1] Some previous works, on the same lines as Corngold's, may also be cited: [9-11].

where $n(\mathbf{r}, \mathbf{v}, t)$ is the number of neutrons per unit volume of the six-dimensional domain $V \otimes \omega$, V and ω (or ω') being, respectively, the body in which the diffusion process occurs, assumed to be a measurable domain of points $\mathbf{r}$, and the space of the velocity vectors $\mathbf{v}$ (or $\mathbf{v}'$), at the time t: $v' \Sigma_s (\mathbf{v}' \to \mathbf{v})$ is the scattering rate per neutron from velocity $\mathbf{v}'$ to velocity $\mathbf{v}$, $v\Sigma_t(v)$ the total collision rate per neutron at velocity $\mathbf{v}$. According to the free gas model, we assume that the nuclei of the scattering medium are free and have a velocity distribution of the Maxwell-Boltzmann type; for the sake of simplicity, we assume also that the scattering cross section of a nucleus, say σ_s, is a constant independent of the relative velocity of the two colliding particles, that inelastic scattering is negligible, and that the total absorption rate per neutron is independent of v (i.e., the macroscopic absorption cross section follows the $1/v$ law). Then, the following relation between the collision rates holds:

$$v\Sigma_t(v) = \int_\omega v\Sigma_s(\mathbf{v} \to \mathbf{v}')\, d\omega' + \gamma \tag{2}$$

where γ is a constant. Finally, we assume that the scattering of the neutrons is isotropic in the laboratory system. This is logically incorrect, since the direction of the emerging neutron is completely determined by the collision laws, which show that the various emerging directions are by no means equally probable. It is known, however, that the assumption of isotropic scattering is the more accurate, the more isotropic is the neutron flux at any point of the body, because averaging over the angles is then permitted, and it is reasonable to suppose that the flux in pulsed experiment samples is not too far from isotropy. Since, moreover, this assumption is unavoidable when solving in practice the eigenvalue problem, we introduce it from the beginning. Explicit expressions for the collision rates can be found in the literature [13-15]: if the new unknown function

$$\psi(\mathbf{r}, \mathbf{v}, t) = e^{-\beta^2 v^2/2}\, n(\mathbf{r}, \mathbf{v}, t) \tag{3}$$

(where $\beta = \sqrt{m/2kT}$, m being the neutron mass, k the Boltzmann constant, and T the temperature of the medium) is introduced, and the independent variables

$$\mathbf{p} = \beta\mathbf{v} \tag{4}$$

$$\tau = \beta^{-1}t \tag{5}$$

are used, then Eq. (1) becomes

$$\frac{\partial\psi(\mathbf{r}, \mathbf{p}, \tau)}{\partial\tau} + \mathbf{p} \cdot \operatorname{grad}_{\mathbf{r}} \psi(\mathbf{r}, \mathbf{p}, \tau) + h(p)\, \psi(\mathbf{r}, \mathbf{p}, \tau) = \int_\omega R(p, p')\, \psi(\mathbf{r}, \mathbf{p}', \tau)\, d\omega \tag{6}$$

where ω, ω' now refer to the new (dimensionless) velocity variables, and

$$h(p) = N\sigma_s \left[\frac{e^{-\mu p^2}}{\sqrt{\pi\mu}} + \left(p + \frac{1}{2\mu p}\right) \operatorname{erf}(\sqrt{\mu} p)\right] + \beta\gamma \tag{7}$$

$$R(p, p') = \frac{S(p, p')}{4\pi p p'} \tag{8}$$

with

$$\begin{aligned} S(p, p') = N\sigma_s \frac{(\mu+1)^2}{4\mu} \Big\{ & e^{(p^2-p'^2)/2} \left[\operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p - \frac{\mu-1}{2\sqrt{\mu}} p'\right)\right. \\ & \qquad \left. \pm \operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p + \frac{\mu-1}{2\sqrt{\mu}} p'\right)\right] \\ & + e^{(p'^2-p^2)/2} \left[\operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p' - \frac{\mu-1}{2\sqrt{\mu}} p\right)\right. \\ & \qquad \left. \mp \operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p' + \frac{\mu-1}{2\sqrt{\mu}} p\right)\right]\Big\}. \end{aligned} \tag{9}$$

Here $\mu = M/m \geqslant 1$ is the ratio of the mass M of a nucleus to the mass of a neutron and N the number of nuclei per unit volume. The upper signs in the last formula refer to $p \leqslant p'$, the lower ones to $p > p'$.

3. The Initial Value Problem

We assume that the body V is finite, convex, and surrounded by the vacuum. The boundary condition to be added to Eq. (6) is then that no neutrons can enter the body from outside; i.e.,

$$\psi(\mathbf{r}, \mathbf{p}, \tau) = 0 \tag{10}$$

when $\mathbf{r}$ is on the boundary Γ of the body and $\mathbf{p}$ points inward.

Now, our problem is to determine the distribution function of the neutrons at any time $\tau > 0$, assuming that the distribution function at the initial instant, say $f_0(\mathbf{r}, \mathbf{p})$, is known:

$$\psi(\mathbf{r}, \mathbf{p}, 0) = f_0(\mathbf{r}, \mathbf{p}). \tag{11}$$

The classical technique is as follows. Let $\mathfrak{H}$ be the Hilbert space of the complex valued functions which are defined and square summable on the domain

$V \otimes \omega$, the norm and the scalar product being defined, as usual, by the formulas:

$$\|f\| = \left\{\int_V\int_\omega |f(\mathbf{r}, \mathbf{p})|^2 \, dV d\omega\right\}^{1/2} \tag{12}$$

$$(f, g) = \int_V\int_\omega f(\mathbf{r}, \mathbf{p}) \overline{g(\mathbf{r}, \mathbf{p})} \, dV d\omega. \tag{13}$$

Equation (6) can be written in the form

$$\frac{\partial \psi}{\partial \tau} = A\psi \tag{14}$$

where A denotes the integrodifferential operator

$$A \cdot = -\mathbf{p} \cdot \mathrm{grad}_{\mathbf{r}} \cdot - h(p) \cdot + \int_\omega R(p, p') \cdot d\omega' \tag{15}$$

defined on the domain $\mathscr{D}_A$ of the functions f in $\mathfrak{H}$ which admit the directional derivative $\mathbf{p} \cdot \mathrm{grad}_{\mathbf{r}} f$, satisfy the boundary condition:

$$f(\mathbf{r}, \mathbf{p}) = 0, \qquad \mathbf{r} \in \Gamma, \ \mathbf{p} \text{ entering the body,} \tag{16}$$

and are such that Af is also in $\mathfrak{H}$. Then, under some very general assumptions, upon which we shall not insist here, one can take the Laplace transform of Eq. (14), solve the nonhomogeneous equation so obtained by means of the resolvent of A, and find the solution of the problem in the form of an inverse Laplace integral

$$\psi(\mathbf{r}, \mathbf{p}, \tau) = \lim_{\eta\to\infty} \frac{1}{2\pi i} \int_{\kappa - i\eta}^{\kappa + i\eta} e^{\lambda\tau} (\lambda I - A)^{-1} f_0(\mathbf{r}, \mathbf{p}) \, d\lambda. \tag{17}$$

Here κ is such that the vertical line $\mathrm{Re}\, \lambda = \kappa$ lies on the right of all the singularities of the resolvent $(\lambda I - A)^{-1}$ (I = identity operator), i.e., on the right of the spectrum of A. By moving the integration path to the left and collecting the residues at the poles of the integrand, i.e., the eigenvalues of A, one can obtain the desired eigenfunction expansion [1-4].

Thus a study of the spectrum σ_A of A is a necessary preliminary, in order to get any information about the existence and the properties of the decay modes.

This paper will be concerned with the spectral analysis only: the passage from the spectral properties to the behavior of the time dependent solution, along the line of the aforementioned works, is now, in the authors' opinion, straightforward (although laborious in the particulars).

4. THE "BAND SPECTRUM" OF A

Let us consider the function $h(p)$. This function has a finite limit as $p \to 0$:

$$\lim_{p\to 0} h(p) = \frac{2N\sigma_s}{\sqrt{\pi\mu}} + \beta\gamma = h_0 > 0. \tag{18}$$

This suffices to prove the following:

THEOREM 1. *The half-plane* $\operatorname{Re}\lambda \leqslant -h_0$ *is contained in* σ_A.

Let us recall a known result ([16], p. 31): if, for a given λ in the spectral plane, a family of functions $u_\delta \in \mathscr{D}_A$ exists such that

$$\| u_\delta \| \geqslant C > 0 \tag{19}$$

$$\| (A - \lambda I) u_\delta \| \to 0 \quad \text{as} \quad \delta \to 0 \tag{20}$$

then λ belongs to σ_A.

Our procedure is reminiscent of that of Lehner and Wing [1]; however, the rôle of their angular variable μ is here played by the neutron speed p.

Let $\mathbf{\Omega} = \mathbf{p}/p$, and let α_Ω be a plane perpendicular to $\mathbf{\Omega}$. Any point $\mathbf{r}$ can be written as follows

$$\mathbf{r} = \mathbf{r}_0 - s\mathbf{\Omega}$$

where $\mathbf{r}_0$ is a point of α_Ω and s a real parameter. Let V_Ω be the projection of V on α_Ω (see Fig. 1). Any point of V can be written, in particular, as $\mathbf{r}_0 - s\mathbf{\Omega}$,

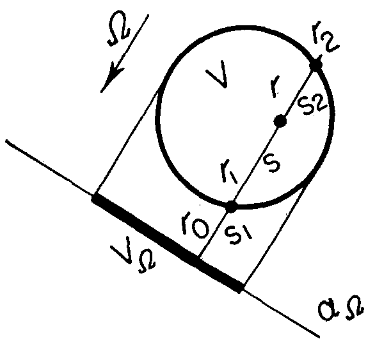

FIG. 1.

where $\mathbf{r}_0$ is in V_Ω and s ranges in the interval $s_1(\mathbf{r}_0, \mathbf{\Omega}) \leqslant s \leqslant s_2(\mathbf{r}_0, \mathbf{\Omega})$, s_1 and s_2 being the values of s for which the line $\mathbf{r} = \mathbf{r}_0 - s\mathbf{\Omega}$ intersects the boundary surface of V. To be definite, we assume that the body V and the plane $\alpha_{\mathbf{\Omega}}$ are placed in such a way that $0 < s_1 \leqslant s_2$. An integral of the type

$$\int_\omega \int_V F(\mathbf{r}, \mathbf{p})\, dV d\omega$$

can be written, accordingly,

$$\int_0^\infty p^2 dp \int_{4\pi} d\Omega \int_{V_{\mathbf{\Omega}}} d\alpha \int_{s_1}^{s_2} F(\mathbf{r}_0 - s\mathbf{\Omega}, p\mathbf{\Omega})\, ds$$

a form which will be of much use in the following.

Let us now consider the following family of functions u_δ, $0 < \delta \leqslant \frac{1}{2}$,

$$u_\delta(\mathbf{r}, \mathbf{p}; \lambda) = e^{(s-s_1)(h_0+\lambda)/p} f(\mathbf{r}, \mathbf{\Omega})\, g(p) \tag{21}$$

where

$$f(\mathbf{r}, \mathbf{\Omega}) = \frac{s_2 - s}{a}$$

$$g(p) = \begin{cases} 1/\delta^2 & \text{if} \quad \delta^2 \leqslant p \leqslant \delta \\ 0 & \text{otherwise} \end{cases}$$

while a is the diameter of V (the maximum distance between any two points of V).

Obviously, $u_\delta \in \mathscr{D}_A$. Also, for any $\beta = \operatorname{Re} \lambda < -h_0$,

$$\begin{aligned} \| u_\delta \|^2 &= \int_\omega \int_V | u_\delta(\mathbf{r}, \mathbf{p}; \lambda) |^2\, dV d\omega \\ &= \int_{\delta^2}^{\delta} p^2 dp \int_{4\pi} d\Omega \int_{V_{\mathbf{\Omega}}} d\alpha \int_{s_1}^{s_2} e^{2(s-s_1)(h_0+\beta)/p} \frac{1}{\delta^4} \left(\frac{s_2 - s}{a}\right)^2 ds. \end{aligned}$$

Let b be a lower bound for the maximum length of the chords drawn in the direction $\mathbf{\Omega}$, as $\mathbf{\Omega}$ varies ($b > 0$ for any well shaped convex body), and let $V_{\mathbf{\Omega}}{}^*$ be the subset of $V_{\mathbf{\Omega}}$ for which $s_2(\mathbf{r}_0, \mathbf{\Omega}) - s_1(\mathbf{r}_0, \mathbf{\Omega}) \geqslant \frac{3}{4} b$. We can assume without difficulty that the body V is such that $V_{\mathbf{\Omega}}{}^*$ is of nonzero measure, for any $\mathbf{\Omega}$. Then

$$\begin{aligned} \| u_\delta \|^2 &\geqslant \frac{b^2}{16 a^2 \delta^4} \int_{\delta^2}^{\delta} p^2 dp \int_{4\pi} d\Omega \int_{V_{\mathbf{\Omega}}{}^*} d\alpha \int_{s_1}^{s_1+(b/2)} e^{2(s-s_1)(h_0+\beta)/p}\, ds \\ &= \frac{b^2}{16 a^2 \delta^4} \int_{4\pi} d\Omega \int_{V_{\mathbf{\Omega}}{}^*} d\alpha \int_{\delta^2}^{\delta} \frac{p^3}{2\, | h_0 + \beta |} (1 - e^{-b|h_0+\beta|/p})\, dp. \end{aligned}$$

Now we assume (having in mind to allow $\delta \to 0$), that δ satisfies the inequality

$$e^{-b|h_0+\beta|/\delta} \leqslant \tfrac{1}{2}.$$

in addition to the previous one $\delta \leqslant \frac{1}{2}$. Thus we get

$$\begin{aligned}\| u_\delta \|^2 &\geqslant \frac{b^2}{64\, a^2\delta^4 \,|\, h_0 + \beta \,|} \int_{4\pi} d\Omega \int_{V_\Omega^*} d\alpha \int_{\delta^2}^{\delta} p^3 dp \\ &= \frac{b^2(1-\delta^4)}{256\, a^2 \,|\, h_0 + \beta} \int_{4\pi} d\Omega \int_{V_\Omega^*} d\alpha \geqslant C_1 > 0\end{aligned}$$

where C_1 depends on β but not on δ.

We have, on the other hand,

$$\begin{aligned}(A - \lambda I)\, u_\delta &= \left[p \frac{\partial\, \cdot}{\partial s} - h(p) \cdot - \lambda \cdot\right] u_\delta(\mathbf{r}, \mathbf{p}; \lambda) + \int_\omega R(p, p')\, u_\delta(\mathbf{r}', \mathbf{p}'; \lambda)\, d\omega' \\ &= z_1 + z_2\end{aligned}$$

where the symbol $p\partial/\partial s$ has been used instead of $-\mathbf{p} \cdot \mathrm{grad}_{\mathbf{r}}$. Now,

$$\begin{aligned}\| z_1 \|^2 = \frac{1}{a^2\delta^4} \int_{\delta^2}^{\delta} p^2 dp \int_{4\pi} d\Omega \int_{V_\Omega} d\alpha \int_{s_1}^{s_2} \\ \times \{[h(p) - h_0]\,(s_2 - s) + p\}^2\, e^{2(s-s_1)(h_0+\beta)/p}\, ds.\end{aligned}$$

It is easily seen (Appendix, (A. 1)) that $h(p) - h_0 \leqslant N\sigma_s p$. Thus

$$\begin{aligned}\| z_1 \|^2 &\leqslant \frac{1}{a^2\delta^4} \int_{\delta^2}^{\delta} p^2 dp \int_{4\pi} d\Omega \int_{V_\Omega} d\alpha \int_{s_1}^{s_2} (N\sigma_s a + 1)^2\, p^2\, e^{2(s-s_1)(h_0+\beta)/p}\, ds \\ &= \frac{(N\sigma_s a + 1)^2}{a^2\delta^4} \int_{4\pi} d\Omega \int_{V_\Omega} d\alpha \int_{\delta^2}^{\delta} \frac{p^5}{2\,|\, h_0 + \beta \,|} \left(1 - e^{-2(s_2-s_1)|h_0+\beta|/p}\right) dp \\ &< \frac{C_2}{|\, h_0 + \beta \,|\, \delta^4} \int_{4\pi} d\Omega \int_{V_\Omega} d\alpha \int_{\delta^2}^{\delta} p^5 dp \\ &< \frac{C_3}{6\,|\, h_0 + \beta \,|} (\delta^2 - \delta^8) \to 0 \qquad \text{as} \qquad \delta \to 0.\end{aligned}$$

Secondly,

$$\| z_2 \|^2 = \int_{4\pi} d\Omega \int_0^\infty p^2 dp \int_V \left| \int_{4\pi} \int_{\delta^2}^{\delta} R(p, p')\, u_\delta(\mathbf{r}, \mathbf{p}'; \lambda)\, p'^2 dp' d\Omega' \right|^2 dV.$$

Now, it can be shown that

$$R(p, p') < \frac{C_4}{4\pi p p'} e^{-C_5 p^2}$$

for $0 < p < \infty$, $0 < p' < \delta \leqslant \frac{1}{2}$ (see (8) and (A. 4)). Then we have, by applying the Schwarz inequality,

$$\begin{aligned}\| z_2 \|^2 &< C_4^2 \int_0^\infty e^{-2C_5 p^2} dp \int_V \left[\int_{\delta^2}^{\delta}\int_{4\pi} | u_\delta(\mathbf{r}, \mathbf{p}'; \lambda) |^2 dp' d\Omega'\right] \\ &\qquad \cdot \left[\int_{\delta^2}^{\delta}\int_{4\pi} p'^2 dp' d\Omega'\right] dV \\ &= \frac{4\pi}{3} C_4^2(\delta^3 - \delta^6) \int_0^\infty e^{-2C_5 p^2} dp \int_{\delta^2}^{\delta} dp' \int_{4\pi} d\Omega' \int_V | u_\delta(\mathbf{r}, \mathbf{p}'; \lambda) |^2 dV \\ &= \frac{4\pi}{3} C_4^2 \frac{\delta^3 - \delta^6}{a^2\delta^4} \int_0^\infty e^{-2C_5 p^2} dp \int_{\delta^2}^{\delta} dp' \int_{4\pi} d\Omega' \int_{V_{\Omega'}} d\alpha \int_{s_1'}^{s_2'} \\ &\qquad \times \exp\left[2(s - s_1') \frac{h_0 + \beta}{p}\right] (s_2' - s)^2 \, ds\end{aligned}$$

where $s_1' = s_1(\mathbf{r}_0, \mathbf{\Omega}')$, $s_2' = s_2(\mathbf{r}_0, \mathbf{\Omega}')$, $\mathbf{r}_0$ being now a point of $V_{\Omega'}$. We obtain, as before,

$$\begin{aligned}\| z_2 \|^2 &< C_6 \frac{\delta^3 - \delta^6}{\delta^4} \int_0^\infty e^{-2C_5 p^2} dp \int_{\delta^2}^{\delta} dp' \int_{4\pi} d\Omega' \int_{V_{\Omega'}} d\alpha \int_{s_1'}^{s_2'} \\ &\qquad \times \exp\left[- 2(s - s_1') \frac{| h_0 + \beta |}{\delta}\right] ds \\ &< C_7 \frac{(\delta - \delta^2)(\delta^3 - \delta^6)}{| h_0 + \beta | \delta^3} \int_{4\pi} d\Omega' \int_{V_{\Omega'}} \frac{\delta}{2 | h_0 + \beta |} \\ &\qquad \times \left\{1 - \exp\left[- 2(s_2' - s_1') \frac{| h_0 + \beta |}{\delta}\right]\right\} d\alpha \\ &< C_8 \delta \frac{(1 - \delta)(1 - \delta^3)}{| h_0 + \beta |} \int_{4\pi} d\Omega' \int_{V_{\Omega'}} d\alpha < \frac{C_9 \delta}{| h_0 + \beta |} \to 0 \quad \text{as} \quad \delta \to 0.\end{aligned}$$

Thus the family u_δ fulfils both requirements (19) and (20) and we conclude that any point λ with $\operatorname{Re} \lambda < - h_0$ is contained in σ_A. But, since σ_A is a closed set, we have that the vertical line $\operatorname{Re} \lambda = - h_0$ is also in σ_A and Theorem 1 is completely proven. We do not seek to investigate if this part of the spectrum is actually contained in the continuous, or residual, or point spectrum, according to the definitions of functional analysis. Whatever may be the type of the spectrum, in fact, the half-plane $\operatorname{Re}\lambda \leqslant -h_0$ is a prohibited domain for the path of integration of the inverse Laplace integral (17), and this implies that, when shifting the path to the left, we must necessarily stop at some $\kappa = -h_0 + \epsilon$, where ϵ is an arbitrarily small positive number. To avoid misunderstandings, the half-plane $\operatorname{Re}\lambda \leqslant -h_0$ will be called henceforth the "band spectrum" of A.

5. The Equivalent Integral Equation

Now we turn our attention to the half-plane $\operatorname{Re}\lambda \geqslant -h_0$[2] and look for the eigenvalues of A which might exist in this domain. The eigenvalue equation $A\varphi = \lambda\varphi$, i.e.,

$$-p\frac{\partial\varphi}{\partial s} + (h + \lambda)\varphi = \int_\omega R(p, p')\,\varphi(\mathbf{r}, \mathbf{p}')\,d\omega' \tag{22}$$

can be integrated along s, as an ordinary differential equation. This gives, if boundary condition (16) is properly accounted for ([12], p. 24):

$$\varphi(\mathbf{r}, \mathbf{p}) = \frac{1}{p}\int_0^{s(\mathbf{r},\boldsymbol{\Omega})} e^{-[(h(p)+\lambda)/p]s}\,ds \int_\omega R(p, p')\,\varphi(\mathbf{r} - s\boldsymbol{\Omega}, \mathbf{p}')\,d\omega' \tag{23}$$

Here, as before, $\boldsymbol{\Omega}$ denotes the unit vector $\mathbf{p}/p$, while $s(\mathbf{r}, \boldsymbol{\Omega})$ is the value of s for which the half-line $\mathbf{r} - s\boldsymbol{\Omega}$, $\mathbf{r} \in V$, $s > 0$, intersects the boundary surface of V.

By integrating both members of (23) on $\boldsymbol{\Omega}$, setting

$$\xi(\mathbf{r}, p) = \int_{4\pi} \varphi(\mathbf{r}, \mathbf{p})\,d\Omega \tag{24}$$

and making use of the following transformation of variables

$$\mathbf{r}' = \mathbf{r} - s\boldsymbol{\Omega}, \qquad dV' = s^2 ds d\Omega \tag{25}$$

we get

$$\xi(\mathbf{r}, p) = \frac{1}{p}\int_V \frac{\exp\left\{-\dfrac{h(p)+\lambda}{p}\,|\,\mathbf{r} - \mathbf{r}'\,|\right\}}{|\,\mathbf{r} - \mathbf{r}'\,|^2}\,dV' \int_0^\infty R(p, p')\,\xi(\mathbf{r}', p')\,p'^2 dp$$

[2] We have just seen that the line $\operatorname{Re}\lambda = -h_0$ belongs to the band spectrum of A. Since, however, many theorems in the following require considering the closed half-plane $\operatorname{Re}\lambda \geqslant -h_0$, better than the open one $\operatorname{Re}\lambda > -h_0$, we shall often extend our investigations to the closure $\operatorname{Re}\lambda = -h_0$.

or also

$$q(\mathbf{r}, p) = \int_V \frac{\exp\left\{-\dfrac{h(p)+\lambda}{p} \mid \mathbf{r} - \mathbf{r}' \mid\right\}}{4\pi p \mid \mathbf{r} - \mathbf{r}' \mid^2} dV \int_0^\infty S(p, p')\, q(\mathbf{r}', p')\, dp' \tag{26}$$

where

$$q(\mathbf{r}, p) = p\xi(\mathbf{r}, p). \tag{27}$$

Hence, if λ is an eigenvalue of A, the integral equation

$$\chi q(\mathbf{r}, p) = \int_V \frac{\exp\left\{-\dfrac{h(p)+\lambda}{p} \mid \mathbf{r} - \mathbf{r}' \mid\right\}}{4\pi p \mid \mathbf{r} - \mathbf{r} \mid^2} dV \int_0^\infty S(p, p')\, q(\mathbf{r}', p')\, dp' \tag{28}$$

will have an eigenvalue equal to one:

$$\chi = \chi(\lambda) = 1. \tag{29}$$

It is easily shown that, conversely, if Eq. (28) has an eigenvalue equal to one, then λ is an eigenvalue of A. We conclude:

LEMMA 1. *Let $P\sigma_A{}^*$ be that part of the point spectrum of A which lies in the half-plane* $\operatorname{Re}\lambda \geqslant -h_0$. *A necessary and sufficient condition that $\lambda \in P\sigma_A{}^*$ is that Eq. (28) has an eigenvalue equal to one. If, in particular,* $\operatorname{Re}\lambda = -h_0$, *this provides a means of determining the kind of this part of the band spectrum.*

We denote by $K(\mathbf{r}, \mathbf{r}'; p, p'; \lambda)$ the kernel of Eq. (28), and by K_λ the corresponding integral operator. If φ is in $\mathfrak{H}$, then

$$\begin{aligned}
\int_V dV \int_0^\infty \mid q(\mathbf{r}, p) \mid^2 dp &= \int_V dV \int_0^\infty \mid \xi(\mathbf{r}, p) \mid^2 p^2 dp \\
&= \int_V dV \int_0^\infty \left| \int_{4\pi} \varphi(\mathbf{r}, \mathbf{p})\, d\Omega \right|^2 p^2 dp \\
&\leqslant \int_V dV \int_0^\infty \left\{ \int_{4\pi} \mid \varphi(\mathbf{r}, \mathbf{p}) \mid^2 d\Omega \cdot \int_{4\pi} d\Omega \right\} p^2 dp \\
&= 4\pi \int_V dV \int_\omega \mid \varphi(\mathbf{r}, \mathbf{p}) \mid^2 d\omega < \infty
\end{aligned} \tag{30}$$

and we see that the natural domain of K_λ is the space $\mathfrak{H}_0$ of the functions square summable on $V \otimes E$, where E is the interval $0 \leqslant p < \infty$. Norm and scalar product in this space will be denoted by $\| \ \|_0$, $(\ ,\ )_0$.

Now, the following very important lemma holds:

LEMMA 2. *If* $\operatorname{Re}\lambda = \beta \geqslant -h_0$, *then* K_λ *is bounded, with*

$$\| K_\lambda \|_0 \leqslant CN\sigma_s a,$$

where C is a constant, depending only on the mass of the nuclei of the medium, and a is the diameter of the body.

Since

$$h(p) + \beta \geqslant h(p) - h_0 \geqslant 0$$

we have

$$\begin{aligned}
&\int_V dV' \int_0^\infty | K(\mathbf{r}, \mathbf{r}'; p, p'; \lambda) | \, dp' \\
&\qquad = \int_V \frac{\exp\left\{-\dfrac{h(p)+\beta}{p} | \mathbf{r} - \mathbf{r}' | \right\}}{4\pi p \, | \mathbf{r} - \mathbf{r}' |^2} dV' \int_0^\infty S(p, p') \, dp' \\
&\qquad = \int_{4\pi} d\Omega \int_0^{s(\mathbf{r},\Omega)} \frac{\exp\left\{-\dfrac{h(p)+\beta}{p} s\right\}}{4\pi p} ds \int_0^\infty S(p, p') \, dp' \\
&\qquad \leqslant \frac{1}{4\pi} \int_{4\pi} d\Omega \int_0^a ds \int_0^\infty \frac{S(p, p')}{p} dp' \\
&\qquad = a \int_0^\infty \frac{S(p, p')}{p} dp' \leqslant CN\sigma_s a
\end{aligned}$$

where use has been made of estimate (A. 5). Similarly, by using (A. 6) we get

$$\begin{aligned}
&\int_V dV \int_0^\infty | K(\mathbf{r}, \mathbf{r}'; p, p'; \lambda) | \, dp \\
&\qquad = \int_V dV \int_0^\infty \frac{\exp\left\{-\dfrac{h(p)+\beta}{p} | \mathbf{r} - \mathbf{r}' | \right\}}{4\pi p \, | \mathbf{r} - \mathbf{r}' |^2} S(p, p') \, dp \\
&\qquad \leqslant a \int_0^\infty \frac{S(p, p')}{p} dp \leqslant CN\sigma_s a.
\end{aligned}$$

Then, if f, g are any two functions in $\mathfrak{H}_0$, with $\| f \|_0 \leqslant 1$, $\| g \|_0 \leqslant 1$, we have

$$\begin{aligned}
| (K_\lambda f, g)_0 | \leqslant \int_V \int_0^\infty \int_V \int_0^\infty & | K(\mathbf{r}, \mathbf{r}'; p, p'; \lambda) | \\
& \cdot | f(\mathbf{r}', p') | \cdot | g(\mathbf{r}, p) | \, dV dp dV' dp'
\end{aligned}$$

$$\leqslant \int_V \int_0^\infty \int_V \int_0^\infty | K(\mathbf{r}, \mathbf{r}'; p, p'; \lambda) |$$
$$\times \{ \tfrac{1}{2} | f(\mathbf{r}', p') |^2 + \tfrac{1}{2} | g(\mathbf{r}, p) |^2 \} \, dV dp dV' dp'$$
$$\leqslant \tfrac{1}{2} C N \sigma_S a \int_V \int_0^\infty | f(\mathbf{r}', p') |^2 \, dV' dp'$$
$$+ \tfrac{1}{2} C N \sigma_S a \int_V \int_0^\infty | g(\mathbf{r}, p) |^2 \, dV dp$$
$$\leqslant C N \sigma_S a,$$

or also ([17], p. 54)

$$\| K_\lambda \|_0 \leqslant C N \sigma_S a, \qquad \text{q.e.d.} \tag{31}$$

Since the modulus of an eigenvalue cannot exceed the norm of the operator, we are led to:

THEOREM 2. *If $a < a_0 = 1/CN\sigma_S$, then no eigenvalue of K_λ can be equal to unity, and $P\sigma_A{}^*$ is a void set.*

Thus we see that, according to Nelkin's idea, no fundamental mode exists if the body is small enough (more precisely, formulas (A. 5) and (A. 6) show that a_0 is of the order of a mean free path).

We can prove more:

LEMMA 3. *If* $\operatorname{Re} \lambda = \beta \geqslant - h_0$, *then K_λ is compact.*

Let us consider the adjoint operator $K_\lambda{}^*$, and form the product $H_\lambda = K_\lambda{}^* K_\lambda$. H_λ is also an integral operator, with the kernel

$$H(\mathbf{r}, \mathbf{r}'; p, p'; \lambda) = \int_V \int_0^\infty S(p'', p) \frac{\exp\left\{ - \dfrac{h(p'') + \bar{\lambda}}{p''} | \mathbf{r}'' - \mathbf{r} | \right\}}{4\pi p'' | \mathbf{r}'' - \mathbf{r} |^2}$$
$$\times \frac{\exp\left\{ - \dfrac{h(p'') + \lambda}{p''} | \mathbf{r}'' - \mathbf{r}' | \right\}}{4\pi p'' | \mathbf{r}'' - \mathbf{r}' |^2} S(p'', p') \, dV'' dp''.$$

Then

$$| H(\mathbf{r}, \mathbf{r}'; p, p'; \lambda) | \leqslant \int_V \frac{1}{4\pi | \mathbf{r} - \mathbf{r}'' |^2} \cdot \frac{1}{4\pi | \mathbf{r}'' - \mathbf{r}' |^2} \, dV''$$
$$\times \int_0^\infty \frac{S(p'', p) \, S(p'', p')}{p''^2} \, dp''$$
$$= I(\mathbf{r}, \mathbf{r}') \cdot J(p, p').$$

But $J(p, p')$ is square summable on $0 \leqslant p,\ p' < \infty$ (see the Appendix, p. 44) and the same will be true of $I(\mathbf{r}, \mathbf{r}')$ on $V \otimes V$, since it is known that a kernel defined on a finite domain, with a singularity of the type $| \mathbf{r} - \mathbf{r}' |^{-2}$, has a square summable first iterate ([18], p. 79). Thus H_λ is an integral operator with a square summable kernel and is therefore compact. By a known theorem ([19], p. 432), K_λ itself is compact.

6. The Resolvent Set ρ_A

We now restrict ourselves to the open half-plane $\operatorname{Re} \lambda > -h_0$. We have seen that this half-plane may contain no eigenvalues of A. In this case, and also in the case in which some eigenvalues exist, it may be asked whether the points $\operatorname{Re} \lambda > -h_0$, which are not eigenvalues, are belonging to the continuous or residual spectrum, or to the resolvent set. Let then Λ be the set

$$\Lambda = \{\lambda \mid \operatorname{Re} \lambda > -h_0,\ \lambda \notin P\sigma_A{}^*\}$$

i.e. the open half-plane $\operatorname{Re} \lambda > -h_0$, deleting any points belonging to $P\sigma_A{}^*$. We shall prove the following theorem:

Theorem 3. *$\Lambda \subset \rho_A$, the resolvent set of A.*

Let us consider the equation

$$(\lambda I - A) f = g, \qquad \lambda \in \Lambda, \qquad g \in \mathfrak{H} \tag{32}$$

i.e., written in full,

$$-p \frac{\partial f}{\partial s} + (h + \lambda) f = \int_\omega R(p, p') f(\mathbf{r}, \mathbf{p}')\, d\omega' + g.$$

Integration along s leads to

$$f(\mathbf{r}, \mathbf{p}) = \frac{1}{p} \int_0^{s(\mathbf{r}, \boldsymbol{\Omega})} e^{-[(h(p)+\lambda)/p]s} \times \left[\int_\omega R(p, p') f(\mathbf{r} - s\boldsymbol{\Omega}, p')\, d\omega' + g(\mathbf{r} - s\boldsymbol{\Omega}, p)\right] ds$$

or also

$$f(\mathbf{r}, p) = \frac{1}{p} \int_0^{s(\mathbf{r}, \boldsymbol{\Omega})} e^{-[(h(p)+\lambda)/p]s} \times \left[\int_0^\infty \frac{S(p, p')}{4\pi p} \zeta(\mathbf{r} - s\boldsymbol{\Omega}, p')\, dp' + g(\mathbf{r} - s\boldsymbol{\Omega}, p)\right] ds \tag{33}$$

where

$$\zeta(\mathbf{r}, p') = p' \int_{4\pi} f(\mathbf{r}, \mathbf{p}') d\Omega'.$$

By integrating both members over $\mathbf{\Omega}$ we get

$$\zeta(\mathbf{r}, p) = \int_V \frac{\exp\left[-\frac{h(p)+\lambda}{p} \mid \mathbf{r} - \mathbf{r}' \mid\right]}{4\pi p \mid \mathbf{r} - \mathbf{r}' \mid^2} \int_0^\infty S(p, p')\, \zeta(\mathbf{r}', p')\, dp' + \eta(\mathbf{r}, p) \tag{34}$$

where

$$\eta(\mathbf{r}, p) = \int_{4\pi} d\Omega \int_0^{s(\mathbf{r},\mathbf{\Omega})} e^{-(h(p)+\lambda/p)s} g(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p})\, ds. \tag{35}$$

Now, η is in $\mathfrak{H}_0$. We have in fact

$$\mid \eta(\mathbf{r}, p) \mid \leqslant \int_{4\pi} d\Omega \int_0^{s(\mathbf{r},\mathbf{\Omega})} e^{-[(h_0+\beta)/p]s} \mid g(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p}) \mid ds$$

whence, using Schwarz' inequality,

$$\begin{aligned}
\mid \eta(\mathbf{r}, p) \mid^2 &\leqslant \int_{4\pi} d\Omega \cdot \int_{4\pi} \left\{ \int_0^{s(\mathbf{r},\mathbf{\Omega})} e^{-[(h_0+\beta)/p]s} \mid g(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p}) \mid ds \right\}^2 d\Omega \\
&= 4\pi \int_{4\pi} \left\{ \int_0^{s(\mathbf{r},\mathbf{\Omega})} e^{-[(h_0+\beta)/2p]s} \mid g(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p}) \mid e^{-[(h_0+\beta)/2p]s} ds \right\}^2 d\Omega \\
&\leqslant 4\pi \int_{4\pi} \left\{ \int_0^{s(\mathbf{r},\mathbf{\Omega})} e^{-[(h_0+\beta)/p]s} ds \right\} \\
&\qquad \cdot \left\{ \int_0^{s(\mathbf{r},\mathbf{\Omega})} e^{-[(h_0+\beta)/p]s} \mid g(\mathbf{r} - s\mathbf{\Omega}, p) \mid^2 ds \right\} d\Omega \\
&\leqslant \frac{4\pi p}{h_0 + \beta} \int_{4\pi} d\Omega \int_0^\infty e^{-[(h_0+\beta)/p]s} \mid g(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p}) \mid^2 ds
\end{aligned}$$

the function $g(\mathbf{r}, \mathbf{p})$ being assumed to be zero if $\mathbf{r} \notin V$. Thus

$$\begin{aligned}
&\int_0^\infty dp \int_V \mid \eta(\mathbf{r}, p) \mid^2 dV \\
&\leqslant \frac{4\pi}{h_0 + \beta} \int_0^\infty p dp \int_V dV \int_{4\pi} d\Omega \int_0^\infty e^{-[(h_0+\beta)/p]s} \mid g(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p}) \mid^2 ds \\
&= \frac{4\pi}{h_0 + \beta} \int_0^\infty p dp \int_{4\pi} d\Omega \int_{\alpha_{\mathbf{\Omega}}} d\alpha \int_{s_1(\mathbf{r}_0,\mathbf{\Omega})}^{s_2(\mathbf{r}_0,\mathbf{\Omega})} ds' \int_0^\infty \\
&\qquad\qquad e^{-[(h_0+\beta)/p]s} \mid g(\mathbf{r}_0 - (s + s')\,\mathbf{\Omega}, p) \mid^2 ds,
\end{aligned}$$

where $\mathbf{r} = \mathbf{r}_0 - s'\mathbf{\Omega}$, $\mathbf{r}_0$ being, as in Section 4, a point of the projection $\alpha_{\mathbf{\Omega}}$ of V on a plane orthogonal to $\mathbf{\Omega}$, and $s_1(\mathbf{r}_0, \mathbf{\Omega})$ $s_2(\mathbf{r}_0, \mathbf{\Omega})$ the intersections of the line $\mathbf{r} = \mathbf{r}_0 - s'\mathbf{\Omega}$ with the surface of V. By interchanging the order of the last two integrations we obtain

$$\int_0^\infty dp \int_V |\eta(\mathbf{r}, p)|^2 \, dV$$

$$\leqslant \frac{4\pi}{h_0 + \beta} \int_0^\infty p dp \int_{4\pi} d\Omega \int_{\alpha_{\mathbf{\Omega}}} d\alpha \int_0^\infty e^{-[(h_0+\beta)/p]s} \, ds$$

$$\times \int_{s_1(\mathbf{r}_0,\mathbf{\Omega})}^{s_2(\mathbf{r}_0,\mathbf{\Omega})} |g(\mathbf{r}_0 - (s + s')\,\mathbf{\Omega}, \mathbf{p})|^2 \, ds'$$

$$\leqslant \frac{4\pi}{h_0 + \beta} \int_0^\infty p dp \int_{4\pi} d\Omega \int_{\alpha_{\mathbf{\Omega}}} d\alpha \int_0^\infty e^{-[(h_0+\beta)/p]s} \, ds \int_{-\infty}^\infty |g(\mathbf{r}_0 - t\mathbf{\Omega}, p)|^2 \, dt$$

$$= \frac{4\pi}{(h_0 + \beta)^2} \int_0^\infty p^2 dp \int_{4\pi} d\Omega \int_{\alpha_{\mathbf{\Omega}}} d\alpha \int_{s_1(\mathbf{r}_0,\mathbf{\Omega})}^{s_2(\mathbf{r}_0,\mathbf{\Omega})} |g(\mathbf{r}_0 - t\mathbf{\Omega}, \mathbf{p})|^2 \, dt$$

$$= \frac{4\pi}{(h_0 + \beta)^2} \int_\omega d\omega \int_V |g(\mathbf{r}, p)|^2 \, dV,$$

so that

$$\|\eta\|_0 \leqslant C_1 \|g\| . \tag{36}$$

Now Eq. (34), i.e., written in short,

$$\zeta = K_\lambda \zeta + \eta, \qquad \eta \in \mathfrak{H}_0 \tag{37}$$

where K_λ is compact on $\mathfrak{H}_0$, is a regular nonhomogeneous integral equation. For $\lambda \in \Lambda$, the number 1 cannot be an eigenvalue of K_λ. Therefore, by Fredholm's alternative theorem, the resolvent $(I - K_\lambda)^{-1}$ exists and is bounded, and we have:

$$\|\zeta\|_0 = \|(I - K_\lambda)^{-1} \eta\|_0 \leqslant \|(I - K_\lambda)^{-1}\|_0 \cdot \|\eta\|_0 < C_2 \|\eta\|_0 . \tag{38}$$

Now, returning to Eq. (33), and putting

$$\nu(\mathbf{r}, p) = \int_0^\infty \frac{S(p, p')}{4\pi p} \zeta(\mathbf{r}, p') \, dp',$$

we obtain, using inequality (A. 7),

$$\begin{aligned}
\| \nu \|^2 &= \int_V dV \int_\omega \left| \int_0^\infty \frac{S(p, p')}{4\pi p} \zeta(\mathbf{r}, p')\, dp' \right|^2 d\omega \\
&= \frac{1}{(4\pi)^2} \int_V dV \int_\omega \left| \int_0^\infty \sqrt{\frac{S(p, p')\, p'}{p}} \cdot \sqrt{\frac{S(p, p')}{pp'}}\, \zeta(\mathbf{r}, p')\, dp' \right|^2 d\omega \\
&\leqslant \frac{1}{(4\pi)^2} \int_V dV \int_\omega \left\{ \int_0^\infty S(p, p') \frac{p'}{p}\, dp' \right\} \cdot \left\{ \int_0^\infty \frac{S(p, p')}{pp'} \mid \zeta(\mathbf{r}, p') \mid^2 dp' \right\} d\omega \\
&\leqslant \frac{C_3}{(4\pi)^2} \int_V dV \int_0^\infty \mid \zeta(\mathbf{r}, p') \mid^2 dp' \int_0^\infty S(p, p') \frac{p}{p'}\, dp \int_{4\pi} d\Omega \\
&\leqslant \frac{C_3^2}{4\pi} \int_V dV \int_0^\infty \mid \zeta(\mathbf{r}, p') \mid^2 dp' \\
&= \frac{C_3^2}{4\pi} \| \zeta \|_0^2.
\end{aligned}$$

Let us set $k = \nu + g$. Then

$$\| k \| \leqslant \| \nu \| + \| g \| \leqslant \frac{C_3}{\sqrt{4\pi}} \| \zeta \|_0 + \| g \| \tag{39}$$

and Eq. (33) can be written

$$f(\mathbf{r}, p) = \frac{1}{p} \int_0^{s(\mathbf{r}, \mathbf{\Omega})} e^{-[(h(p)+\lambda)/p]s}\, k(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p})\, ds.$$

Now we can see that f is a bounded element of $\mathfrak{H}$. In fact, setting $k(\mathbf{r}, p) = 0$ when $\mathbf{r} \notin V$, we have:

$$\begin{aligned}
\| f \|^2 &= \int_{4\pi} d\Omega \int_0^\infty dp \int_{\alpha_\mathbf{\Omega}} d\alpha \int_{s_1(\mathbf{r}_0, \mathbf{\Omega})}^{s_2(\mathbf{r}_0, \mathbf{\Omega})} \\
&\qquad \times \left| \int_0^{s(\mathbf{r}_0 - s'\mathbf{\Omega}, \mathbf{\Omega})} e^{-[(h(p)+\lambda)/p]s}\, k(\mathbf{r} - s\mathbf{\Omega}, \mathbf{p})\, ds \right|^2 ds' \\
&\leqslant \int_{4\pi} d\Omega \int_0^\infty dp \int_{\alpha_\mathbf{\Omega}} d\alpha \int_{s_1(\mathbf{r}_0, \mathbf{\Omega})}^{s_2(\mathbf{r}_0, \mathbf{\Omega})} \left\{ \int_0^\infty e^{-[(h_0+\beta)/p]s}\, ds \right\} \\
&\qquad \cdot \left\{ \int_0^\infty e^{-[(h_0+\beta)/p]s} \cdot \mid k(\mathbf{r}_0 - (s + s')\, \mathbf{\Omega}, \mathbf{p}) \mid^2 ds \right\} ds' \\
&\leqslant \frac{1}{(h_0 + \beta)^2} \int_{4\pi} d\Omega \int_0^\infty p^2 dp \int_{\alpha_\mathbf{\Omega}} d\alpha \int_{-\infty}^\infty \mid k(\mathbf{r}_0 - t\mathbf{\Omega}, \mathbf{p}) \mid^2 dt \\
&= \frac{1}{(h_0 + \beta)^2} \int_\omega d\omega \int_V \mid k(\mathbf{r}, \mathbf{p}) \mid^2 dV \\
&= \frac{1}{(h_0 + \beta)^2} \| k \|^2.
\end{aligned} \tag{40}$$

By collecting inequalities (36), (38), (39), (40) we get

$$\| f \| \leqslant \frac{1}{h_0 + \beta} \left\{ \frac{C_3}{\sqrt{4\pi}} C_2 C_1 + 1 \right\} \| g \| < C_4 \| g \| \tag{41}$$

for any $g \in \mathfrak{H}$, $\lambda \in \Lambda$. Therefore $(\lambda I - A)^{-1}$, $\lambda \in \Lambda$, is a bounded operator whose domain is the entire space $\mathfrak{H}$ and our theorem is proved.

7. The Point Spectrum $P\sigma_A^{**}$

Let us now consider the set $P\sigma_A^{**} = \{\lambda \mid \lambda \in P\sigma_A{}^*, \operatorname{Re} \lambda > -h_0\}$. We have see that $P\sigma_A{}^*$, and consequently $P\sigma_A^{**}$, the set of the eigenvalues of A not contained in the band spectrum, can be void, for very small bodies. The following analysis is directed to determine the structure of this part of the point spectrum, in the case that the body is not so small that the existence of the eigenvalues is excluded a priori. We return to Eq. (26)

$$q = K_\lambda q, \qquad \operatorname{Re} \lambda > -h_0 \tag{42}$$

where K_λ can be written

$$K_\lambda = G_\lambda S \tag{43}$$

with

$$G_\lambda f = \int_V \frac{\exp\left[-\dfrac{h(p) + \lambda}{p} \mid \mathbf{r} - \mathbf{r}' \mid \right]}{4\pi p \mid \mathbf{r} - \mathbf{r}' \mid} f(\mathbf{r}', p)\, dV' \tag{44}$$

$$Sf = \int_0^\infty S(p, p') f(\mathbf{r}, p')\, dp'. \tag{45}$$

S is a bounded symmetric operator [20], G_λ is symmetric when λ is real. Suppose that λ ($\operatorname{Re} \lambda > -h_0$) is such that Eq. (42) has a nontrivial solution $q \neq 0$, i.e., λ is an eigenvalue of A. Then $(q, Sq)_0 = (G_\lambda Sq, Sq)_0 = (G_\lambda \theta, \theta)_0$ where $\theta = Sq$. Thus we see that

$$(G_\lambda \theta, \theta)_0 = \int_0^\infty dp \int_V \int_V \frac{\exp\left[-\dfrac{h(p) + \lambda}{p} \mid \mathbf{r} - \mathbf{r}' \mid \right]}{4\pi p \mid \mathbf{r} - \mathbf{r}' \mid^2} \theta(\mathbf{r}', p)\, \overline{\theta(\mathbf{r}, p)}\, dV dV' \tag{46}$$

must be real. Following the technique first developed by Lehner and Wing ([1], p. 222), we extend the definition of $\theta(\mathbf{r}, p)$ to all space, by assuming $\theta(\mathbf{r}, p) = 0$ if $\mathbf{r}$ is not in V; the double volume integration in (46) can thus be extended over all space V_∞.

Then we consider the Fourier transform

$$\int_{V_\infty} \frac{\exp\left[-\dfrac{h(p)+\lambda}{p} r + i\mathbf{w}\cdot\mathbf{r}\right]}{4\pi p r^2} dV = \frac{1}{pw} \arctan \frac{pw}{h(p)+\lambda} \tag{47}$$

([12], p. 52; this formula has been derived for $h(p)+\lambda$ real and positive, but it holds for any $\operatorname{Re}\lambda > -h_0$, by analytical continuation). By applying the convolution theorem, identity (46) is transformed into the following one:

$$(G_\lambda \theta, \theta)_0 = \int_0^\infty dp \int_{V_\infty} \frac{1}{pw} \arctan \frac{pw}{h(p)+\lambda} \cdot |\tilde{\theta}(\mathbf{w}, p)|^2 \, dV_{\mathbf{w}} \tag{48}$$

where $\tilde{\theta}$ is the Fourier transform of θ. It is easily seen that the imaginary part of $(1/pw) \arctan\{pw/[h(p)+\lambda]\}$ is permanently positive or zero or negative as $\operatorname{Im}\lambda$ is negative or zero or positive ([1], p. 223). Thus λ must be a real number, since otherwise $(G_\lambda\theta, \theta)_0$ could not be real (the alternative $\theta = Sq = 0$ is readily excluded, because $Sq = 0$ would imply $q = 0$, by (42)). We conclude:

THEOREM 4. *The half-plane* $\operatorname{Re}\lambda > -h_0$ *contains no complex (not real) eigenvalues of* A.

We can therefore restrict ourselves to the real λ's, namely, to those belonging to the interval $(-h_0, \infty)$. To avoid confusion we write β instead of λ, when λ is real. From (48) we get, using the inequality $(1/x) \arctan x \leqslant 1$ $(0 \leqslant x < \infty)$, and applying Parseval identity,

$$0 \leqslant (G_\beta \theta, \theta)_0 \leqslant \int_0^\infty dp \int_{V_\infty} \frac{1}{h(p)+\beta} |\tilde{\theta}(\mathbf{w}, p)|^2 \, dV_{\mathbf{w}} \leqslant \frac{1}{h_0+\beta} \|\theta\|_0^2,$$
$$\theta \in \mathfrak{H}_0,$$

hence $\|G_\beta\|_0 \to 0$ as $\beta \to +\infty$ and, consequently, $\|K_\beta\|_0 \to 0$, $\beta \to +\infty$. This means (Lemma 1) that the eigenvalues of A must be less than some number $\beta_M > -h_0$, or

THEOREM 5. $P\sigma_A^{**}$ *is contained in a finite interval* $-h_0 \leqslant \beta < \beta_M$ *of the real axis.*

We still need a pair of lemmas, before arriving at our final goal:

LEMMA 4. *The family* K_β *is continuous in the norm on* $-h_0 \leqslant \beta < \infty$.

Let $-h_0 \leqslant \beta < \beta' < \infty$ and consider the operator

$$\Delta_{\beta,\beta'} = (K_\beta{}^* - K_{\beta'}{}^*)(K_\beta - K_{\beta'}),$$

whose kernel is

$$\Delta(\mathbf{r}, \mathbf{r}'; p, p'; \beta, \beta') = \int_V \int_0^\infty \frac{S(p'', p)\, S(p'', p')}{p''^2} \cdot \frac{\exp\left\{ - \frac{h(p'') + \beta}{p''} (|\mathbf{r}'' - \mathbf{r}| + |\mathbf{r}'' - \mathbf{r}'|) \right\}}{16\pi^2 |\mathbf{r}'' - \mathbf{r}|^2 |\mathbf{r}'' - \mathbf{r}'|^2} \cdot \left(1 - \exp\left(- \frac{\beta' - \beta}{p''} |\mathbf{r}'' - \mathbf{r}|\right)\right) \times \left(1 - \exp\left\{- \frac{\beta' - \beta}{p''} |\mathbf{r}'' - \mathbf{r}'|\right\}\right) dV'' dp'' \geqslant 0.$$

We see, just as for the operator H_λ in Lemma 3, that $\Delta(\mathbf{r}, \mathbf{r}'; p, p'; \beta, \beta')$ is square summable. Now we consider the functions $F_l(x) = 1 - e^{-l/x}$, $0 \leqslant x < \infty$, $l > 0$, which form an ordered set, since $F_l(x) \leqslant F_{l'}(x)$ if $l < l'$. This set converges to the function $F_0(x) = 0$ $(0 < x < \infty)$, $F_0(0) = 1$, as $l \to 0$.

Setting $l = \beta' - \beta$, $x = p''/|r'' - r|$, or $x = p''/|r'' - r'|$, we find, by the aid of Levi's theorem, that

$$N^2_{\beta,\beta'} \equiv \int_V \int_0^\infty \int_V \int_0^\infty \Delta^2(\mathbf{r}, \mathbf{r}'; p, p'; \beta, \beta')\, dV dp dV' dp' \to 0 \text{ as } \beta' \to \beta, \beta' > \beta.$$

A similar proof holds for $-h_0 < \beta' < \beta < \infty$. Then we have ([19], pp. 206, 177),

$$\| K_\beta - K_{\beta'} \|_0^2 = \| \Delta_{\beta,\beta'} \|_0 \leqslant N_{\beta,\beta'} \to 0 \qquad \text{as} \qquad \beta' \to \beta, \qquad \text{q.e.d.}$$

LEMMA 5. *The family $SK_\beta = SG_\beta S$ is differentiable in the norm with respect to β on $-h_0 \leqslant \beta < \infty$; its derivative D_β is a self-adjoint compact operator and has the property $(D_\beta f, f)_0 < 0$ if $Sf \neq 0, f \in \mathfrak{H}_0$.*

Let $-h_0 \leqslant \beta' < \beta < \infty$ and consider the kernel of the operator $S[(G_\beta - G_{\beta'})/(\beta - \beta')]\, S$ i.e.,

$$\Xi(\mathbf{r}, \mathbf{r}'; p, p'; \beta, \beta') = - \int_0^\infty \frac{S(p, p'')\, S(p'', p')}{p''^2} \cdot \frac{\exp\left\{ - \frac{h(p'') + \beta}{p''} |\mathbf{r} - \mathbf{r}'| \right\}}{4\pi |\mathbf{r} - \mathbf{r}'|} \cdot \frac{p''}{(\beta' - \beta) |\mathbf{r} - \mathbf{r}'|} \times \left(1 - \exp\left\{- \frac{\beta' - \beta}{p''} |\mathbf{r} - \mathbf{r}'|\right\}\right) dp''. \tag{49}$$

We observe that the functions $\Phi_l(x) = (x/l)\,(1 - e^{-l/x})$, $0 \leqslant x < \infty$, $l > 0$, converge to the function $\Phi_0(x) = 1$ $(0 < x < \infty)$, $\Phi_0(0) = 0$. Moreover, $\Phi_0(x) \geqslant \Phi_l(x)$. Setting $l = \beta' - \beta$, $x = p''/|\,\mathbf{r} - \mathbf{r}'\,|$, and recalling that $\int_0^\infty (1/p'')^2\, S(p, p'')\, S(p'', p')\, dp''$ is square summable on p, p' (Appendix, p. 44) we see that the kernel $\Xi(\mathbf{r}, \mathbf{r}'; p, p'; \beta, \beta')$ itself is square summable.

Next, letting $\beta' \to \beta$, we obtain the kernel

$$D(\mathbf{r}, \mathbf{r}'; p, p'; \beta) = -\int_0^\infty \frac{S(p, p'')\, S(p'', p')}{p''^2} \cdot \frac{\exp\left\{-\dfrac{h(p'') + \beta}{p''}\,|\,\mathbf{r} - \mathbf{r}'\,|\right\}}{4\pi\,|\,\mathbf{r} - \mathbf{r}'\,|}\, dp'' \tag{50}$$

which will be also square summable. But, as inequality $\Phi_0(x) \geqslant \Phi_l(x)$ implies $D^2(\mathbf{r}, \mathbf{r}'; p, p'; \beta) \geqslant \Xi^2(\mathbf{r}, \mathbf{r}'; p, p'; \beta, \beta')$, we get, by the Lebesgue "dominated convergence" theorem, that $D(\mathbf{r}, \mathbf{r}'; p, p'; \beta)$ is also the limit in the mean of $\Xi(\mathbf{r}, \mathbf{r}'; p, p'; \beta, \beta')$. The same proof holds, with slight changes, if $-h_0 < \beta' < \beta < \infty$. Since double-norm convergence implies ordinary ($\mathfrak{H}_0$) norm convergence, we conclude that the self-adjoint compact operator D_β, defined by (50), is the limit in the norm of the operator $S[(G_\beta - G_{\beta'})/(\beta - \beta')]\, S$, or also D_β is the uniform derivative of $SG_\beta S$. We have, in addition, $(D_\beta f, f)_0 < 0$ for $Sf \neq 0$: the Fourier transform of the function $(4\pi p''^2 r)^{-1} \exp\{-[h(p'') + \beta]\, r/p''\}$ is equal, in fact, to

$$1/\{[h(p'') + \beta]^2 + p''^2 w^2\},$$

for $-h_0 \leqslant \beta < \infty$, and we get, by a procedure similar to that of Theorem 4,

$$(D_\beta f, f)_0 = -\int_0^\infty dp'' \int_{V_\infty} \frac{1}{[h(p'') + \beta]^2 + p''^2 w^2}\, |\,\tilde{\theta}(\mathbf{w}, p)\,|^2\, dV_{\mathbf{w}}$$

where $\tilde{\theta}(\mathbf{w}, p)$ is the Fourier transform of $\theta = Sf$, $f \in \mathfrak{H}_0$, with $\theta(\mathbf{r}, p) = 0$ if $\mathbf{r}$ is outside V. If $\theta = Sf \neq 0$, the right hand member is negative, q.e.d.

We have finally:

Theorem 6. *$P\sigma_A^{**}$ has no accumulation points.*

This proof could be considered as the extension to continuous velocity dependence of a proof of Pimbley ([3], p. 852) obtained originally within the framework of multigroup theory. Suppose that β^*, $-h_0 \leqslant \beta^* < \beta_M$, is an accumulation point of eigenvalues of A, $\beta_n \to \beta^*$, say.

Owing to the continuity of K_β as a function of β, the eigenvalues of K_{β_n} converge to the eigenvalues of K_{β^*}, while the corresponding eigenspaces of K_{β_n} converge to the eigenspaces of K_{β^*} ([21], p. 1091).

In particular, we can select a sequence of eigenfunctions q_n corresponding to the eigenvalues $\chi_i(K_{\beta_n}) = 1$, i.e., $q_n = K_{\beta_n} q_n$, in such a way that $q_n \to q^*$ in the norm, where q^* is an eigenfunction of the limit problem $q^* = K_{\beta^*} q^*$.[3]

Obviously $Sq^* \neq 0$. Next, let us consider the following identity

$$(K_{\beta_n} - K_{\beta^*})\, q_n = q_n - K_{\beta^*} q_n$$

or also, after multiplication by $(\beta_n - \beta^*)^{-1}$,

$$\frac{G_{\beta_n} - G_{\beta^*}}{\beta_n - \beta^*}\, S q_n = \frac{1}{\beta_n - \beta^*} (I - G_{\beta^*} S)\, q_n \,.$$

Now,

$$\begin{aligned} ((I - G_{\beta^*} S)\, q_n \,, Sq^*)_0 &= ((S - SG_{\beta^*} S)\, q_n \,, q^*)_0 \\ &= (Sq_n \,, q^*)_0 - (q_n \,, SG_{\beta^*} Sq^*)_0 \\ &= (Sq_n \,, q^*)_0 - (q_n \,, Sq^*)_0 = 0. \end{aligned}$$

It follows,

$$\left(\frac{G_{\beta_n} - G_{\beta^*}}{\beta_n - \beta^*}\, Sq_n \,, Sq^*\right)_0 = \left(S\, \frac{G_{\beta_n} - G_{\beta^*}}{\beta_n - \beta^*}\, Sq_n \,, q^*\right)_0 = 0.$$

This equation, in the limit $n \to \infty$, would lead to $(D_{\beta^*} q^*, q^*)_0 = 0$, with $Sq^* \neq 0$, a result which we know to be false (Lemma 5). Our theorem is thus proved.

From Theorems 5 and 6 we conclude, by virtue of the Bolzano-Weierstrass theorem:

THEOREM 7. *$P\sigma_A^{**}$ is made of finitely many isolated points.*

8. THE SPECTRAL THEOREM

We collect Theorem 1 to 6 and obtain:

MAIN THEOREM. *The linear operator A, from $\mathfrak{H}$ to itself, decomposes the spectral plane λ as follows:*

(a) *The half-plane* $\operatorname{Re} \lambda \leqslant -h_0$ *is contained in* σ_A *("band spectrum").*

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[3] The theorem quoted above deals only with the eigenvalues; the convergence of the eigenspaces, however, follows from classical arguments: in our case, $\| K_{\beta^*} - K_{\beta_n} \| \to 0$ implies $q_n - K_{\beta^*} q_n \to 0$, and the compactness of K_{β^*} enforces—let us suppose the subsequence already extracted—$q_n \to q^*$, $K_{\beta^*} q_n \to q^*$, and, finally, $K_{\beta^*} q^* = q^*$.

(b) *Those points in the half-plane* $\operatorname{Re} \lambda > -h_0$ *which are not contained in* $P\sigma_A$ *are belonging to* ρ_A.

(c) *Those points in the half-plane* $\operatorname{Re} \lambda > -h_0$ *which are contained in* $P\sigma_A$ *form the set* $P\sigma_A^{**}$. $P\sigma_A^{**}$ *contains at most a finite number of points which lie on a finite interval of the real axis,* $-h_0 < \beta < \beta_M$. *If the diameter of the body* V *is smaller than some constant* $a_0 > 0$ *(of the order of a mean free path), then* $P\sigma_A^{**}$ *is a void set, and no eigenvalue exists.*

Thus we can foresee that the solution of the initial value problem proposed in Section 3, under the approximation involved by the free gas model and the isotropy of the scattering, should be written as a finite sum of discrete modes (when existing, i.e., if the body is not very small), plus a continuum which does not seem to allow for a better representation than that of the inverse Laplace integral. It is then necessary that the size of the body is such that at least one discrete eigenvalue exists, in order that a meaningful theoretical analysis of pulsed neutron experiments is possible.

APPENDIX

We derive some estimates concerning the functions $h(p)$ and $S(p, p')$. We have, from (7),

$$\lim_{p \to 0} h(p) = h_0 = \frac{2N\sigma_S}{\sqrt{\pi\mu}} + \beta\gamma > 0$$

$$\lim_{p \to +\infty} \frac{1}{p} h(p) = N\sigma_S > 0$$

$$h'(p) = N\sigma_S \cdot \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\mu}p} \left(1 - \frac{t^2}{\mu p^2}\right) e^{-t^2} \, dt > 0.$$

Hence

$$\begin{aligned} 0 \leqslant h(p) - h_0 &= N\sigma_S \left[\frac{e^{-\mu p^2} - 2}{\sqrt{\pi\mu}} + p \operatorname{erf}(\sqrt{\mu}p) + \frac{1}{2\mu p} \operatorname{erf}(\sqrt{\mu}p)\right] \\ &< N\sigma_S \left[\frac{e^{-\mu p^2} - 2}{\sqrt{\pi\mu}} + p + \frac{1}{\sqrt{\pi\mu}p} \int_0^{\sqrt{\mu}p} dt\right] \\ &= N\sigma_S \left[\frac{e^{-\mu p^2} - 1}{\sqrt{\pi\mu}} + p\right] \leqslant N\sigma_S p. \end{aligned} \tag{A.1}$$

Then we recall that $S(p, p')$ in a nonnegative, symmetric function whose explicit form is

$$S(p,p') = N\sigma_s \frac{(\mu+1)^2}{4\mu} \Big\{ \exp\left(\frac{p^2 - p'^2}{2}\right) \Big[\operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p - \frac{\mu-1}{2\sqrt{\mu}} p'\right) \pm \operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p + \frac{\mu-1}{2\sqrt{\mu}} p'\right) \Big]$$
$$+ \exp\left(\frac{p'^2 - p^2}{2}\right) \Big[\operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p' - \frac{\mu-1}{2\sqrt{\mu}} p\right) \mp \operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p' + \frac{\mu-1}{2\sqrt{\mu}} p\right) \Big] \Big\}$$

for $p \lessgtr p'$ respectively. If $p < p'$ the last term in square brackets is negative, so that we get

$$S(p,p') < N\sigma_s \frac{(\mu+1)^2}{4\mu} \exp\left(\frac{p^2 - p'^2}{2}\right) \Big[- \operatorname{erf}\left(- \frac{\mu+1}{2\sqrt{\mu}} p + \frac{\mu-1}{2\sqrt{\mu}} p'\right) + \operatorname{erf}\left(\frac{\mu+1}{2\sqrt{\mu}} p + \frac{\mu-1}{2\sqrt{\mu}} p'\right) \Big]$$

where we have made use of the identity $\operatorname{erf}(x) = -\operatorname{erf}(-x)$. This inequality can also be written

$$S(p,p') < N\sigma_s \frac{(\mu+1)^2}{4\mu} \exp\left(\frac{p^2 - p'^2}{2}\right) \frac{2}{\sqrt{\pi}} \int_{-[(\mu+1)/2\sqrt{\mu}]p+[(\mu-1)/2\sqrt{\mu}]p}^{[(\mu+1)/2\sqrt{\mu}]p+[(\mu-1)/2\sqrt{\mu}]p'} e^{-t^2}\, dt$$

and we get, replacing e^{-t^2} by one,

$$S(p,p') < N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} p \exp\left(\frac{p^2 - p'^2}{2}\right) \qquad (p < p') \qquad \text{(A.2)}$$

In a similar way we obtain

$$S(p,p') < N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} p' \exp\left(\frac{p'^2 - p^2}{2}\right) \qquad (p > p'). \qquad \text{(A.3)}$$

Thus, if $0 \leqslant p < \infty$, $0 \leqslant p' < \delta \leqslant \frac{1}{2}$ the following inequality holds

$$S(p,p') < Ce^{-c'p^2}. \qquad \text{(A.4)}$$

Furthermore, by using the above inequalities we get:

$$\begin{aligned} I_1 &= \int_0^\infty \frac{S(p,p')}{p}\, dp' \\ &< N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} \left\{ \int_0^p \exp\left(\frac{p'^2-p^2}{2}\right) dp' + \int_p^\infty \exp\left(\frac{p^2-p'^2}{2}\right) dp' \right\} \end{aligned}$$

where the factor p'/p in the first integral of the second member has been replaced by 1. It follows

$$\begin{aligned} I_1 &< N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} \left\{ \int_0^p \exp\left(-\frac{(p+p')(p-p')}{2}\right) dp' \right. \\ &\qquad\qquad\qquad\qquad \left. + 2\int_0^\infty e^{-y^2} \frac{y\,dy}{\sqrt{p^2+2y^2}} \right\} \\ &< N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} \left\{ \int_0^p \exp\left(-\frac{p(p-p')}{2}\right) dp' + \sqrt{2}\int_0^\infty e^{-y^2}\, dy \right\} \\ &= N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} \left\{ \frac{2}{p}(1-e^{-p^2/2}) + \sqrt{\frac{\pi}{2}} \right\} \\ &\leqslant N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} \left(1+\sqrt{\frac{\pi}{2}}\right) = CN\sigma_s \end{aligned} \tag{A.5}$$

since it is easily seen that the function $(2/p)\,[1-\exp(-p^2/2)]$ cannot exceed the unity. Similarly

$$\begin{aligned} I_2 &= \int_0^\infty \frac{S(p,p')}{p}\, dp \\ &< N\sigma_s \frac{(\mu+1)^3}{2\mu\sqrt{\pi\mu}} \left\{ \int_0^{p'} \exp\left(\frac{p^2-p'^2}{2}\right) dp + \int_{p'}^\infty \exp\left(\frac{p'^2-p^2}{2}\right) dp \right\} \\ &< CN\sigma_s\,. \end{aligned} \tag{A.6}$$

We consider also another kernel,

$$J(p,p') = J(p',p) = \int_0^\infty \frac{S(p,p'')\,S(p',p'')}{p''^2}\, dp''.$$

By using repeatedly (A.2) and (A.3) we easily get the estimate

$$\begin{aligned} J(p,p') &= C\int_0^\infty \exp\left(-\tfrac{1}{2}\,|\,p'^2-p''^2\,|\right) \frac{S(p,p'')}{p''}\, dp'' \\ &\leqslant C^2 \int_0^\infty \exp\left(-\tfrac{1}{2}\,|\,p^2-p''^2\,|\right) \exp\left(-\tfrac{1}{2}\,|\,p'^2-p''^2\,|\right) dp'' \\ &= C^2 L(p,p'). \end{aligned}$$

We show that $L(p, p')$, and consequently $J(p, p')$, is square summable. Owing to the symmetry of this kernel, it will be sufficient to show that the integral

$$\iint_T L^2(p, p')\, dp dp'$$

converges, T being the triangle $0 \leqslant p \leqslant p'$, $0 \leqslant p' < \infty$. We set

$$L(p, p') = L_1(p, p') + L_2(p, p') + L_3(p, p')$$

where

$$L_1(p, p') = \int_0^p \exp\left[-\tfrac{1}{2}(p^2 - p''^2)\right] \exp\left[-\tfrac{1}{2}(p'^2 - p''^2)\right] dp''$$

$$L_2(p, p') = \int_p^{p'} \exp\left[-\tfrac{1}{2}(p''^2 - p^2)\right] \exp\left[-\tfrac{1}{2}(p'^2 - p''^2)\right] dp''$$

$$L_3(p, p') = \int_{p'}^{\infty} \exp\left[-\tfrac{1}{2}(p''^2 - p^2)\right] \exp\left[-\tfrac{1}{2}(p''^2 - p'^2)\right] dp''.$$

The Minkowsky inequality allows us to write

$$\left\{\iint_T L^2 dp dp'\right\}^{1/2} \leqslant \left\{\iint_T L_1^2 dp dp'\right\}^{1/2} + \left\{\iint_T L_2^2 dp dp'\right\}^{1/2} + \left\{\iint_T L_3^2 dp dp'\right\}^{1/2}$$

so that we need only establish the existence of the last three integrals. Now, by the Schwarz inequality, and using the fact that $[1 - \exp(-p^2)]/p \leqslant 1/\sqrt{2}$ we have

$$\begin{aligned} L_1^2(p, p') &\leqslant \int_0^p \exp\left[-(p + p'')(p - p'')\right] dp'' \cdot \int_0^p \exp\left[-(p' + p'')(p' - p'')\right] dp'' \\ &\leqslant \int_0^p e^{-px}\, dx \int_{p'-p}^{p'} e^{-p'y}\, dy = \frac{1}{p}(1 - e^{-p^2}) \cdot \frac{e^{-p'^2}}{p'}(e^{pp'} - 1) \\ &\leqslant \frac{e^{-p'^2}}{\sqrt{2}\, p'}(e^{pp'} - 1). \end{aligned}$$

Thus

$$\begin{aligned} \iint_T L_1^2 dp dp' &\leqslant \int_0^{\infty} \frac{e^{-p'^2}}{\sqrt{2}\, p'}\, dp' \int_0^{p'} (e^{pp'} - 1)\, dp \\ &= \frac{1}{\sqrt{2}} \int_0^{\infty} \left(\frac{1 - e^{-p'^2}}{p'^2} - e^{-p'^2}\right) dp' < \infty. \end{aligned}$$

Next

$$L_2^2(p, p') = (p' - p)^2 e^{-(p'^2 - p^2)}$$

$$\iint_T L_2^2 dp dp' = \int_0^\infty dp' \int_0^{p'} (p' - p)^2 \exp[-(p' + p)(p' - p)]\, dp$$

$$\leqslant \int_0^\infty dp' \int_0^{p'} x^2 e^{-p'x}\, dx$$

$$= \int_0^\infty \left(2 \frac{1 - e^{-p'^2} - p'^2 e^{-p'^2}}{p'^3} - p' e^{-p'^2}\right) dp' < \infty.$$

Finally,

$$L_3(p, p') = \int_{p'}^\infty \exp\left[-\left(p''^2 - \frac{p^2 + p'^2}{2}\right)\right] dp''$$

$$\leqslant \exp\left(-\frac{p'^2 - p^2}{2}\right) \int_0^\infty \frac{e^{-x^2}\, x dx}{\sqrt{p'^2 + x^2}}$$

having set $p''^2 = p'^2 + x^2$. Two inequalities can be derived from the preceeding one:

$$L_3(p, p') \leqslant \exp\left(-\frac{p'^2 - p^2}{2}\right) \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \exp\left(-\frac{p'^2 - p^2}{2}\right)$$

$$L_3(p, p') \leqslant \frac{1}{p'} \exp\left(-\frac{p'^2 - p^2}{2}\right) \int_0^\infty e^{-x^2}\, x dx = \frac{1}{2p'} \exp\left(-\frac{p'^2 - p^2}{2}\right).$$

It follows:

$$\iint_T L_3^2 dp dp' \leqslant \frac{\pi}{4} \int_0^1 dp' \int_0^{p'} \exp[-(p'^2 - p^2)]\, dp + \frac{1}{4} \int_1^\infty \frac{dp'}{p'^2} \int_0^{p'} \exp[-(p'^2 - p^2)]\, dp$$

$$< \frac{\pi}{4} \int_0^1 dp' \int_0^{p'} dp + \frac{1}{4} \int_1^\infty \frac{dp'}{p'^2} \int_0^{p'} \exp[-(p' + p)(p' - p))\, dp$$

$$< \frac{\pi}{8} + \frac{1}{4} \int_1^\infty \frac{dp'}{p'^2} \int_0^{p'} \exp[-p'(p' - p)]\, dp$$

$$< \frac{\pi}{8} + \frac{1}{4} \int_1^\infty \frac{dp'}{p'^2} \int_0^\infty e^{-p'x}\, dx = \frac{\pi + 1}{8} < \infty.$$

Our assertion is thus completely proved.

We shall prove yet another inequality:

$$\int_0^\infty S(p, p') \frac{p'}{p} dp' \leqslant C < \infty. \tag{A.7}$$

The best way to see this is to recall, from thermal neutron diffusion theory [13-15], that $S(p, p')$ is obtained from the complete free gas scattering kernel $H(\mathbf{p}, \mathbf{p}')$ (see below) through an integration over the angles:

$$S(p, p') = pp' \int_{4\pi} H(\mathbf{p}, \mathbf{p}') \, d\Omega'$$

$$= pp' \cdot N\sigma_s \frac{(\mu + 1)^2}{4(\pi\mu)^{3/2}} \int_{4\pi} \frac{1}{P} \exp\left\{ - \tfrac{1}{4} \left[\frac{P^2}{\mu} + \frac{\mu}{P^2} (p^2 - p'^2)^2 \right] \right\} d\Omega'$$

where $P = |\mathbf{p} - \mathbf{p}'|$. This formula may also be verified directly. Then

$$\int_0^\infty S(p, p') \frac{p'}{p} dp' = \int_\omega H(\mathbf{p}, \mathbf{p}') \, d\omega' < C \int_0^\infty P e^{-p^2/4\mu} \, dP < \infty \qquad \text{q.e.d.}$$

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References

1. J. Lehner and G. M. Wing. On the spectrum of an asymmetric operator arising in the transport thery of neutrons. *Commun. Pure Appl. Math.* **8** (1955), 217-234.
2. J. Lehner and G. M. Wing. Solution of the linearized Boltzmann equation for the slab geometry. *Duke Math. J.* **23** (1956), 125-142.
3. G. Pimbley. Solution of an initial value problem for the multi-velocity neutron transport equation with a slab geometry. *J. Math. Mech.* **8** (1959), 837-866.
4. K. Jőrgens. An asymptotic expansion in the theory of neutron transport. *Commun. Pure Appl. Math.* **11** (1958), 219-242.
5. G. M. Wing. "An Introduction to Transport Theory." Wiley, New York, 1962.
6. R. Van Norton. "The Spectrum of a Neutron Transport Operator." Rept., NYO 9085 (New York Univ. 1960).
7. R. S. Nelkin. Asymptotic solutions of the transport equation for thermal neutrons. *Physica* **29** (1963), 261-273.
8. N. Corngold. Some transient phenomena in thermalization. *Nucl. Sci. Eng.* **19** (1964), 80-90.
9. K. M. Case. Elementary solutions of the transport equation and their applications. *Ann. Phys. (N.Y.)* **9** (1960), 1-23.
10. J. Koppel. A method of solving the time dependent neutron thermalization problem. *Nucl. Sci. Eng.* **16** (1963), 101-110.

11. N. CORNGOLD, P. MICHAEL, AND W. WOLLMAN. The time decay constants in neutron thermalization. *Nucl. Sci. Eng.* **15** (1963), 13-19.
12. B. DAVISON. "Neutron Transport Theory." Oxford Univ. Press, 1957.
13. E. R. COHEN. "A Survey of Neutron Thermalization Theory." PIGC P/611, Vol. 5, pp. 405-416 (1955).
14. G. JAFFÉ. Diffusion of neutrons. *Phys. Rev.* **88** (1952), 603-611.
15. R. K. OSBORN. Some characteristics of the thermal neutron scattering probability. *Nucl. Sci. Eng.* **3** (1957), 29-37.
16. E. HILLE AND R. S. PHILLIPS. Functional analysis and semigroups. *Am. Math. Soc. Colloq. Publ.* **31** (1957).
17. M. H. STONE. Linear transformations in Hilbert space, *Am. Math. Soc. Colloq. Publ.* **15** (1932).
18. S. G. MIKHLIN. "Linear Integral Equations," Hindustan, 1960.
19. A. C. ZAANEN. "Linear Analysis." North Holland, Amsterdam, 1960.
20. B. MONTAGNINI AND M. L. DEMURU. Complete continuity of the free gas scattering operator in neutron thermalization theory. *J. Math. Anal. Appl.*, in press.
21. N. DUNFORD AND J. T. SCHWARTZ. "Linear Operators," Vols. I, II. Interscience, New York, 1964.